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# Asymptotic solutions of coupled dynamic problems of thermoelasticity for isotropic plates☆

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ABSTRACT

The asymptotic method of solving boundary-value problems of the theory of elasticity for anisotropic strips and plates is used to solve coupled dynamic problems of thermoelasticity for plates, on the faces of which the values of the temperature function and the values of the components of the displacement vector or the conditions of the mixed problem of the theory of elasticity are specified. Recurrence formulae are derived for determining the components of the displacement vector, the stress tensor and for the temperature field variation function of the plate.



The asymptotic method proposed earlier[1,2] turned out to be effective for solving problems of the theory of elasticity both with static[1–7] and dynamic[8–11] boundary conditions. Problems of thermoelasticity for anisotropic laminated plates and shells were solved by this method[3–7] on the assumption that the temperature field variation function satisfies the heat-conduction equation.[12] The solution of coupled dynamic problems of thermoelasticity[13] for plates is urgent.

## 1. Formulation of the boundary-value problems and the derivation of the resolvents

Consider a plate which, in a rectangular system of coordinates, occupies the region

$$\Omega_* = \{x, y, z: -a \le x \le a, -b \le y \le b, -h \le z \le h, h \ll l = \min\{a, b\}\}$$

(see Fig. 1).

The change in temperature $\theta = T - T_0$ and the components of the displacement vector $u = (u_x, u_y, u_z)$

$$\theta(x, y, -h, t) = \theta^-(x, y, t), \quad u_j(x, y, -h, t) = u_j^-(x, y, t), \quad j = x, y, z \tag{1.1}$$

are given on the surface $z = -h$, while on the opposite surface $z = h$, the change in temperature

$$\theta(x, y, +h, t) = \theta^+(x, y, t) \tag{1.2}$$

is given together with the components of the displacement vector

$$u_j(x, y, +h, t) = u_j^+(x, y, t), \quad j = x, y, z \tag{1.3}$$

or with the components of the stress tensor

$$\sigma_{jz}(x, y, h, t) = \sigma_{jz}^+(x, y, t), \quad j = x, y, z \tag{1.4}$$

or with one of the combinations of the following mixed conditions

$$u_z(x, y, h, t) = u_z^+(x, y, t), \quad \sigma_{jz}(x, y, h, t) = \sigma_{jz}^+(x, y, t), \quad j = x, y \tag{1.5}$$

or

$$u_j(x, y, h, t) = u_j^+(x, y, t), \quad j = x, y; \quad \sigma_{zz}(x, y, h, t) = \sigma_{zz}^+(x, y, t) \tag{1.6}$$

It is required to determine the stress-strain state of the plate and the change in its temperature field, assuming the process to be steady, so that the initial conditions are not specified. The boundary conditions on the ends $x = \pm a$, $y = \pm b$ of the plate are not given, since an internal problem is being solved. To solve this problem it is necessary to obtain the solution of the following system of equations[13]

$$G\nabla^2 u_x + \frac{G}{1-2\nu}\frac{\partial}{\partial x}\operatorname{div}\mathbf{u} + X = \gamma^*\frac{\partial\theta}{\partial x} + \rho\ddot{u}_x \ (x, y, z; X, Y, Z)$$
$$\nabla^2\theta - \frac{1}{\chi}\dot{\theta} - \eta^*\frac{\partial}{\partial t}\operatorname{div}\mathbf{u} = -\frac{P}{\chi} \tag{1.7}$$

which satisfy boundary conditions (1.1) and (1.2) and one of the versions of conditions (1.3)–(1.6).

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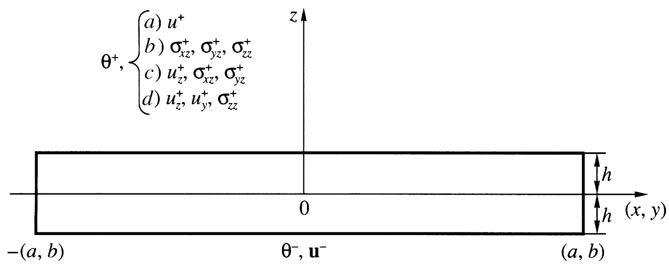


**Fig. 1.** Versions *a-d* in the figure correspond to conditions (1.3)–(1.6).

Here

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad \gamma^* = \frac{2\alpha^* G(1+\nu)}{1-2\nu},$$

$$\eta^* = \gamma^* \frac{T_0}{\lambda^*}, \quad \chi = \frac{\lambda^*}{c_\varepsilon}, \quad P = \frac{W^*}{\lambda^*}$$

G is the shear modulus, $\nu$ is Poisson's ratio, ρ is the density, *X*, *Y* and *Z* are the components of the body forces, $\chi$ is the thermal diffusivity, $\lambda^*$ is the thermal conductivity, $c_\varepsilon$ is the specific heat capacity at constant strain, $\alpha^*$ is the coefficient of linear expansion, $W^*$ is the heat source specific density and $T_0$ is the initial absolute temperature.

We will rewrite the relation between the components of the stress tensor, the displacement vector and the change in the temperature in the form of the Duhamel–Neumann law[13]

$$\sigma_{xx} = 2G\frac{\partial u_x}{\partial x} + \frac{2\nu G}{1-2\nu}\operatorname{div}\mathbf{u} - \gamma^*\theta, \quad \sigma_{xy} = G\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \ (x, y, z) \tag{1.8}$$

Suppose the specified functions in boundary conditions (1.1)–(1.6) and in Eq. (1.7) have the form

$$U(x, y, \pm h, t) = U^{\pm}(x, y)\sin\omega t, \quad U = \{u_j, \sigma_{jz}\}, \quad j = x, y, z \tag{1.9}$$

$$X(x, y, z, t) = X_1(x, y, z)\sin\omega t \ (X, Y, Z, P) \tag{1.10}$$

It is then more convenient to represent all the required quantities in relations (1.7) and (1.8) in the form

$$Q(x, y, z, t) = Q_1(x, y, z)\sin\omega t + Q_2(x, y, z)\cos\omega t \ (Q, \theta) \tag{1.11}$$

where Q is any of the components of the displacement tensor, the stress tensor, and also the temperature field variation function.

Substituting expressions (1.10) and (1.11) into Eq. (1.7) and changing to dimensionless coordinates and displacement using the formulae

$$\xi = \frac{x}{l}, \ \eta = \frac{y}{l}, \ \zeta = \frac{z}{h} = \varepsilon^{-1}\frac{z}{l}, \ \varepsilon = \frac{h}{l},$$

$$u_k = \frac{u_{xk}}{l}, \ \upsilon_k = \frac{u_{yk}}{l}, \ w_k = \frac{u_{zk}}{l}, \ k = 1, 2 \tag{1.12}$$

we obtain

$$G\tilde{\nabla}_1^2 u_k + \frac{G}{1-2\nu}\frac{\partial}{\partial\xi}(\tilde{\nabla}_1\cdot\mathbf{u}_k) + lX_k = \gamma^*\frac{\partial\theta_k}{\partial\xi} - \varepsilon^{-2}\omega^2\rho h^2 u_k$$

$$(\xi, \eta, u_k, \upsilon_k, X_k, Y_k), \quad k = 1, 2, \quad X_2 = Y_2 = 0$$

$$G\tilde{\nabla}_1^2 w_k + \frac{G}{1-2\nu}\varepsilon^{-1}\frac{\partial}{\partial\zeta}(\tilde{\nabla}_1\cdot\mathbf{u}_k) + lZ_1 = \gamma^*\varepsilon^{-1}\frac{\partial\theta_k}{\partial\zeta} - \varepsilon^{-2}\omega^2\rho h^2 w_k,$$

$$k = 1, 2$$

$$\tilde{\nabla}_\varepsilon^2\theta_1 + \varepsilon^{-2}\frac{\omega h^2}{\chi}\theta_2 + \varepsilon^{-2}\omega\eta^* h^2(\tilde{\nabla}_\varepsilon\cdot\mathbf{u}_2) = -l^2\frac{P_1}{\chi}$$

$$\tilde{\nabla}_\varepsilon^2\theta_2 - \varepsilon^{-2}\frac{\omega h^2}{\chi}\theta_1 - \varepsilon^{-2}\omega\eta^* h^2(\tilde{\nabla}_\varepsilon\cdot\mathbf{u}_1) = 0 \tag{1.13}$$

Here

$$\tilde{\nabla}_1^2 = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\zeta^2}, \quad \tilde{\nabla}_\varepsilon^2 = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} + \varepsilon^{-2}\frac{\partial^2}{\partial\zeta^2}$$

$$(\tilde{\nabla}_1\mathbf{u}_k) = \frac{\partial u_k}{\partial\xi} + \frac{\partial\upsilon_k}{\partial\eta} + \frac{\partial w_k}{\partial\zeta}, \quad (\tilde{\nabla}_\varepsilon\mathbf{u}_k) = \frac{\partial u_k}{\partial\xi} + \frac{\partial\upsilon_k}{\partial\eta} + \varepsilon^{-1}\frac{\partial w_k}{\partial\zeta}$$

The system of equations (1.13) is singularly perturbed by the geometric small parameter $\varepsilon$. Its solution is obtained by matching the solution of the internal problem with the solution of the problem for a boundary layer. It was proved in Refs 3, 7, 14, 15 that the solution of the internal problem predominates at internal points of the plate, beginning at a distance of (1.5–2)h from its side surface. The solution of the internal problem on the side surface of the plate in general ceases to predominate in view of the fact that here the effect of the boundary-layer solution is greater, and it attenuates exponentially in the direction of the inward normal to the side surface.

The solution of the internal problem will be sought in the form of the asymptotic expansion

$$Q_k(x, y, z) = \sum_{s=0}^{S}\varepsilon^{\chi_Q}Q_k^{(s)}(\xi, \eta, \zeta), \quad k = 1, 2 \tag{1.14}$$

where $Q_k$ is any of the unknown components of the displacement vector $u_j$, the stress tensor $\sigma_{ij}$ and also the change in the temperature $\theta$, and $\chi_Q$ is the asymptotic order of the corresponding quantity, where for all the displacements $\chi_u = 0$, for all the stresses $\chi_\sigma = -1$, and for the temperature field variation function $\chi_\theta = -1$.[1–7] These asymptotic orders were introduced for the first time in Refs 1 and 2 for boundary-value problems of the theory of strips and plates with kinematic and mixed boundary conditions, similar to conditions (1.1)–(1.6).

We will represent the specified body forces and the normalized specific density of the heat source in the form of the following asymptotic expansions

$$X_1(x, y, z) = \sum_{s=0}^{S} l^{-1}\varepsilon^{s-2}X_1^{(s)}(\xi, \eta, \zeta) \ (X, Y, Z),$$

$$P_1(x, y, z) = \sum_{s=0}^{S} l^{-2}\varepsilon^{s-3}P_1^{(s)}(\xi, \eta, \zeta) \tag{1.15}$$

This indicates that the body forces and the heat source can affect the stress-strain state, beginning with the first step of the iteration process, if their asymptotic orders are $\varepsilon^{-2}$ and $\varepsilon^{-3}$ respectively. Substituting expressions (1.14) and (1.15) into relations (1.13) and equating the coefficients of $\varepsilon^s$ ($s=0,1,2,\ldots,S$) on the left and right sides of the equations, we obtain a non-contradictory system of second-order differential equations with constant coefficients in the unknown expansion coefficients (1.14). As a result, we obtain the system of resolvents in the form

$$\frac{\partial^2 u_k^{(s)}}{\partial\zeta^2}+\gamma^2 u_k^{(s)} = R_{uk}^{(s)}\ (u_k, \upsilon_k)$$

$$R_{uk}^{(s)} = -\frac{1}{G}X_k^{(s)}+\frac{\gamma^*}{G}\frac{\partial\theta_k^{(s-1)}}{\partial\xi}-\frac{1}{1-2\nu}\frac{\partial^2 w_k^{(s-1)}}{\partial\xi\partial\zeta}-\frac{2(1-\nu)}{1-2\nu}\frac{\partial^2 u_k^{(s-2)}}{\partial\eta^2}-\frac{1}{1-2\nu}\frac{\partial^2\upsilon_k^{(s-2)}}{\partial\xi\partial\eta}$$

$$(\xi, \eta; u_k, \upsilon_k; X_k, Y_k);\quad k = 1, 2,\quad X_2^{(s)} = Y_2^{(s)} = 0$$

$$\frac{\partial^2 w_k^{(s)}}{\partial\zeta^2}+pw_k^{(s)} = \beta^*\frac{\partial\theta_k^{(s)}}{\partial\zeta}+R_{wk}^{(s)} \tag{1.16}$$

$$R_{wk}^{(s)} = -\frac{1-2\nu}{2G(1-\nu)}Z_k^{(s)}-\frac{1}{2(1-\nu)}\left(\frac{\partial^2 u_k^{(s-1)}}{\partial\xi\partial\zeta}+\frac{\partial^2\upsilon_k^{(s-1)}}{\partial\eta\partial\zeta}\right)-\frac{1-2\nu}{2(1-\nu)}\left(\frac{\partial^2 w_k^{(s-2)}}{\partial\xi^2}+\frac{\partial^2 w_k^{(s-2)}}{\partial\eta^2}\right)$$

$$\frac{\partial^2\theta_k^{(s)}}{\partial\zeta^2}+(-1)^{3-k}q\theta_{3-k}^{(s)}+r\frac{\partial w_{3-k}^{(s)}}{\partial\zeta} = R_{\theta 1}^{(s)}$$

$$R_{\theta_k}^{(s)} = -\delta_{1k}\frac{P_1^{(s)}}{\chi}-(-1)^{3-k}r\left(\frac{\partial u_{3-k}^{(s-1)}}{\partial\xi}+\frac{\partial\upsilon_{3-k}^{(s-1)}}{\partial\eta}\right)-\frac{\partial^2\theta_k^{(s-2)}}{\partial\xi^2}-\frac{\partial^2\theta_k^{(s-2)}}{\partial\eta^2};$$

$$k = 1, 2,\ \delta_{11} = 1,\ \delta_{12} = 0 \tag{1.17}$$

Here

$$\gamma = \omega h\sqrt{\frac{\rho}{G}},\quad p = \gamma^2\frac{1-2\nu}{2(1-\nu)},\quad q = \frac{\omega h^2}{\chi},$$

$$r = \omega\eta^* h^2,\quad \beta^* = \alpha^*\frac{1+\nu}{1-\nu}$$

Taking relations (1.11), (1.12) and (1.14) into account we write the components of the stress tensor (1.8) in the form

$$\sigma_{xxk}^{(s)} = \frac{2\nu G}{1-2\nu}\frac{\partial w_k^{(s)}}{\partial\zeta}-\gamma^*\theta_k^{(s)}+R_{xxk}^{(s)}\ (x, y)$$

$$R_{xxk}^{(s)} = \frac{2(1-\nu)G}{1-2\nu}\frac{\partial u_k^{(s-1)}}{\partial\xi}+\frac{2\nu G}{1-2\nu}\frac{\partial\upsilon_k^{(s-1)}}{\partial\eta}\ (x, y; \xi, \eta; u, \upsilon)$$

$$\sigma_{zzk}^{(s)} = \frac{2(1-\nu)G}{1-2\nu}\frac{\partial w_k^{(s)}}{\partial\zeta}-\gamma^*\theta_k^{(s)}+R_{zzk}^{(s)};$$

$$R_{zzk}^{(s)} = \frac{2\nu G}{1-2\nu}\left(\frac{\partial u_k^{(s-1)}}{\partial\xi}+\frac{\partial\upsilon_k^{(s-1)}}{\partial\eta}\right)$$

$$\sigma_{xyk}^{(s)} = G\left(\frac{\partial u_k^{(s-1)}}{\partial\eta}+\frac{\partial\upsilon_k^{(s-1)}}{\partial\xi}\right),\ \sigma_{xzk}^{(s)} = G\left(\frac{\partial u_k^{(s)}}{\partial\zeta}+\frac{\partial w_k^{(s-1)}}{\partial\xi}\right),$$

$$\sigma_{yzk}^{(s)} = G\left(\frac{\partial\upsilon_k^{(s)}}{\partial\zeta}+\frac{\partial w_k^{(s-1)}}{\partial\eta}\right);\ k = 1, 2 \tag{1.18}$$

Hence, it is necessary to obtain the solution of system of equations (1.16), (1.17) which satisfies one of the combinations of boundary conditions (1.1)–(1.6) taking representation (1.11) into account.

## 2. Solutions of the boundary-value problems

System (1.16) consists of four equations, which are independent in a first approximation, i.e., they can be coupled only after the first step of the iteration. We will call them quasi-coupled equations. Their general solutions have the form

$$u_k^{(s)} = M_{uk}^{(s)}\sin\gamma\zeta+N_{uk}^{(s)}\cos\gamma\zeta+J_{uk}^{(s)}(\zeta),$$

$$J_{uk}^{(s)} = \frac{1}{\gamma}\int_0^\zeta R_{uk}^{(s)}(\tau)\sin\gamma(\zeta-\tau)d\tau(u, \upsilon);\ k = (1, 2) \tag{2.1}$$

System (1.17) consists of four equations, related to the first step of the iteration. Its solution has the form

$$w_k^{(s)} = \sum_{n=1}^{8}C_n^{(s)}A_{wk}(\lambda_n)e^{\lambda_n\zeta}+J_{wk}^{(s)}(\zeta),$$

$$\theta_k^{(s)} = \sum_{n=1}^{8}C_n^{(s)}A_{\theta k}(\lambda_n)e^{\lambda_n\zeta}+J_{\theta k}^{(s)}(\zeta);\ k = 1, 2 \tag{2.2}$$

Here $C_n^{(s)}$ are functions of integration, $\lambda_n$ ($n=1,2,\ldots,8$) are the roots of the equation

$$\begin{vmatrix} \lambda^2+p & \lambda^2+p & -\beta^*\lambda & -\beta^*\lambda \\ -r\lambda & 0 & -q & \lambda^2 \\ 0 & r\lambda & \lambda^2 & q \\ 0 & \lambda^2+p & 0 & -\beta^*\lambda \end{vmatrix} = 0 \tag{2.3}$$

which have the form

$$\lambda_n = \pm\sqrt{-(p+(-1)^n ic)\pm\sqrt{p^2-c^2+2(-1)^n ip(c-2q)}}/\sqrt{2}$$

$$n = 1, 2, \ldots, 8,\quad c = q+r\beta^* \tag{2.4}$$

(all eight possible combinations of signs must be taken into account), $A_{wk}$, $A_{\theta k}$ ($k=1, 2$) are the cofactors of the elements of the first row of determinant (2.3)

$$A_{w1}(\lambda_n) = -(\lambda_n^2+p)(\lambda_n^4+q^2)-\lambda_n^2 qr\beta^*,\quad A_{w2}(\lambda_n) = -\lambda_n^4 r\beta^*$$

$$A_{\theta 1}(\lambda_n) = \lambda_n^3 rc+\lambda_n pqr,\quad A_{\theta 2}(\lambda_n) = -\lambda_n^3 r(\lambda_n^2+p) \tag{2.5}$$

and $J_{wk}^{(s)}(\xi), J_{\theta k}^{(s)}(\xi) (k=1, 2)$ are particular solutions of the system of coupled inhomogeneous (with non-zero right-hand side) equations (1.17), which have the form

$$J_{wk}^{(s)}(\zeta) = \sum_{n=1}^{8}C_n^{*(s)}(\zeta)A_{wk}(\lambda_n)e^{\lambda_n\zeta}(w, \theta),\quad k = 1, 2 \tag{2.6}$$

where $C_n^{*(s)}(\xi)$ are the elements of the column matrices

$$\mathrm{col}[C_1^{*(s)}(\zeta), C_2^{*(s)}(\zeta), \ldots, C_8^{*(s)}(\zeta)] = \int_0^\zeta\|a_{ij}\|_{8\times 8}^{-1}\mathrm{col}[R_{w_1}^{(s)}, R_{w2}^{(s)}, R_{\theta 1}^{(s)}, R_{\theta 2}^{(s)}, 0, \ldots, 0]d\zeta$$

$$a_{mn} = \lambda_n a_{(m+4)n},\quad m = 1, 2, 3, 4$$

$$a_{(k+4)n} = A_{wk}(\lambda_n)e^{\lambda_n\zeta},\quad a_{(k+6)n} = A_{\theta k}(\lambda_n)e^{\lambda_n\zeta};$$

$$k = 1, 2,\quad n = 1, 2, \ldots, 8 \tag{2.7}$$

The general solutions (2.1) and (2.2) contain 16 functions of integration

$$M_{uk}^{(s)}, N_{uk}^{(s)}(u, \upsilon); \quad k = 1, 2; \quad C_n^{(s)}, \quad n = 1, 2, \ldots, 8$$

which are defined uniquely from the boundary conditions, taking transformations (1.9) into account.

By satisfying the boundary conditions (1.1)–(1.3), we obtain eight functions of integration

$$\begin{aligned}
&2M_{uk}^{(s)}\sin\gamma = u_k^{+(s)} - L_{uk}^{(s)}, \quad 2N_{uk}^{(s)}\cos\gamma = u_k^{+(s)} + L_{uk}^{(s)}\\
&L_{uk}^{(s)} = u_k^{-(s)} + J_{uk}^{(s)}(\zeta = 1) - J_{uk}^{(s)}(\zeta = -1) \ \ (u, \upsilon),\\
&k = 1, 2, \quad u_2^{\pm(s)} = \upsilon_2^{\pm(s)} = 0\\
&u_1^{\pm(0)} = \frac{u_x^{\pm}}{l}, \quad u_k^{\pm(s)} = 0, \quad s > 0 \ \ (x, y, u, \upsilon), \quad \sin 2\gamma \neq 0
\end{aligned} \tag{2.8}$$

We obtain the remaining eight functions of integration from relations (1.1)–(1.3) and (2.2)

$$\begin{aligned}
&\operatorname{col}[C_1^{(s)}, C_2^{(s)}, \ldots, C_8^{(s)}] = \|b_{ij}\|_{8\times 8}^{-1}\operatorname{col}[F_1^{+(s)}, F_1^{-(s)}, F_2^{+(s)}, F_2^{-(s)},\\
&F_3^{+(s)}, F_3^{-(s)}, F_4^{+(s)}, F_4^{-(s)}]\\
&b_{(4k-3)n} = A_{wk}(\lambda_n)e^{\lambda_n}, b_{(4k-2)n} = A_{wk}(\lambda_n)e^{-\lambda_n}, b_{(4k-1)n} = A_{\theta k}(\lambda_n)e^{\lambda_n},\\
&b_{(4k)n} = A_{\theta k}(\lambda_n)e^{-\lambda_n}\\
&k = 1, 2; \quad n = 1, 2, \ldots, 8\\
&F_{2k-1}^{\pm(s)} = \delta_{1k}w_k^{\pm(s)} - J_{wk}^{(s)}(\zeta = \pm 1), \quad F_{2k}^{\pm(s)} = \delta_{1k}\theta_k^{+(s)} - J_{\theta k}^{(s)}(\zeta = \pm 1),\\
&\delta_{11} = 1, \quad \delta_{12} = 0\\
&w_1^{\pm(0)} = \frac{u_{z1}^{\pm}}{l}, \quad w_1^{\pm(s)} = 0, \quad s > 0, \quad \det\|b_{ij}\|_{8\times 8} \neq 0
\end{aligned} \tag{2.9}$$

Hence, the recurrence formulae (2.1)–(2.9), together with relations (1.14), (1.9)–(1.11) and (1.8) enable us to calculate the components of the displacement vector, the stress tensor and the temperature field inside the plate with any asymptotic accuracy $O(\varepsilon^s)$, when we are given conditions (1.1)–(1.3) on the faces.

By satisfying boundary conditions (1.1), (1.2) and (1.4), taking relations (1.19), (1.11) and (1.18) into account, we obtain

$$\begin{aligned}
&M_{uk}^{(s)}\cos 2\gamma = I_{xzk}^{(s)}\cos\gamma + I_{uk}^{(s)}\sin\gamma,\\
&N_{uk}^{(s)}\cos 2\gamma = I_{uk}^{(s)}\cos\gamma + I_{xzk}^{(s)}\sin\gamma; \quad k = 1, 2\\
&I_{xzk}^{(s)} = \left[\sigma_{xzk}^{+(s)} - \left(\frac{\partial}{\partial\zeta}J_{uk}^{(s)}(\zeta) + G\frac{\partial w_k^{(s-1)}}{\partial\xi}\right)\right]_{\zeta = 1},\\
&I_{uk}^{(s)} = u_k^{-(s)} - J_{uk}^{(s)}(\zeta = -1)\\
&(\xi, \eta; x, y; u, \upsilon)\\
&\sigma_{xz1}^{+(0)} = \sigma_{xz1}^{\pm}, \quad \sigma_{xz1}^{(s)} = 0, \quad s > 0 \ \ (x, y),\\
&\sigma_{xz2}^{+(s)} = \sigma_{yz2}^{+(s)} = u_2^{-(s)} = \nu_2^{-(s)} = 0
\end{aligned} \tag{2.10}$$

and we write the values of the remaining functions of integration in the form of a matrix

$$\begin{aligned}
&\operatorname{col}[C_1^{(s)}, C_2^{(s)}, \ldots, C_8^{(s)}] = \|d_{ij}\|_{8\times 8}^{-1}\operatorname{col}[\Phi_1^{(s)}, \Phi_1^{(s)},\\
&\Phi_3^{(s)}, \Phi_4^{(s)}, \Phi_5^{+(s)}, \Phi_5^{-(s)}, \Phi_6^{+(s)}, \Phi_6^{-(s)}]\\
&d_{(2k-1)n} = \left(\frac{2(1-\nu)G}{1-2\nu}A_{wk}(\lambda_n) - \gamma^* A_{\theta k}(\lambda_n)\right)e^{\lambda_n},\\
&d_{(2k)n} = A_{wk}(\lambda_n)e^{-\lambda_n}\\
&d_{(2k+3)n} = A_{\theta k}(\lambda_n)e^{\lambda_n}, \quad d_{(2k+4)n} = A_{\theta k}(\lambda_n)e^{-\lambda_n}\\
&\Phi_{2k-1}^{(s)} = \delta_{1k}\sigma_{zzk}^{+(s)} - J_{zzk}^{(s)}(\zeta = 1), \quad \Phi_{2k}^{(s)} = \delta_{1k}u_{zk}^{-(s)} - J_{wk}^{(s)}(\zeta = -1)\\
&\Phi_{k+4}^{\pm(s)} = \delta_{1k}\theta_k^{\pm(s)} - J_{\theta k}^{(s)}(\zeta = \pm 1), \quad \delta_{11} = 1, \quad \delta_{12} = 0\\
&J_{zzk}^{(s)} = \frac{2(1-\nu)G}{1-2\nu}\frac{\partial}{\partial\zeta}J_{wk}^{(s)}(\zeta) - \gamma^* J_{\theta k}^{(s)}(\zeta) + R_{zzk}^{(s)}; \quad k = 1, 2\\
&\sigma_{zz1}^{+(0)} = \sigma_{zz1}^{+}, \quad \sigma_{zz1}^{+(s)} = 0, \quad s > 0, \quad \sigma_{zz2}^{+(s)} = u_{z2}^{-(s)} = 0\\
&u_{z1}^{-(0)} = \frac{u_z^{-}}{l}, \quad u_{zk}^{-(s)} = 0, \quad s > 0; \quad k = 1, 2, \quad \det\|d_{ij}\|_{8\times 8} \neq 0
\end{aligned} \tag{2.11}$$

For mixed boundary conditions (1.1), (1.2) and (1.5) the functions of integration are given by formulae (2.9) and (2.10), while for conditions (1.1), (1.2) and (1.6) they are given by formulae (2.8) and (2.11).

The above boundary-value problems have unique solutions when

$$\sin 2\gamma \neq 0, \quad \cos 2\gamma \neq 0, \quad \det\|b_{ij}\|_{8\times 8} \neq 0, \quad \det\|d_{ij}\|_{8\times 8} \neq 0 \tag{2.12}$$

If at least one of these conditions is not satisfied, thermal-wave resonance occurs. Here, the principal values which lead to resonance of the frequencies of natural vibrations

$$\omega_{\text{res}} = \frac{\pi n}{2h}\sqrt{\frac{G}{\rho}}, \quad \omega_{\text{res}} = \frac{\pi n}{4h}\sqrt{\frac{G}{\rho}} \tag{2.13}$$

and are determined from the equations $\sin 2\gamma = 0$, $\cos 2\gamma = 0$, give rise to resonance of the shear vibrations (2.1) with boundary conditions (1.1)–(1.3) and (1.1), (1.2), (1.4) respectively. These resonance frequencies are independent of the thermal coefficients of the plate material. At the same time, the principal values of the resonance frequencies, determined by the dispersion equations $\det||b_{ij}||_{8\times 8} = 0$, $\det||d_{ij}||_{8\times 8} = 0$, are closely related to the thermal effects immediately from the first step of the iteration; they can be determined only when the specific physical-mechanical constants of the plate are taken into account. Note that the recurrence formulae (1.4), (1.18) and (2.1)–(2.11) derived enable one to obtain analytical solutions of the boundary-value problems in question with any asymptotic accuracy using a computer in a few minutes. The solutions are obtained with limitations (1.9)–(1.11), imposed on the boundary conditions (1.1)–(1.6) and on the functions specified in Eq. (1.7).

These limitations can be avoided by replacing all the functions specified in relations (1.1)–(1.6) and (1.7) by Fourier transforms:

$$Q^{\pm}(x, y, z, t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} Q_1^{\pm}(x, y, z, \omega)\sin\omega t\, d\omega$$

$$Q_1^{\pm}(x, y, z, \omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} Q^{\pm}(x, y, z, t)\sin\omega t\, dt$$

$$(Q^{\pm}, X, Y, Z, P; Q_1^{\pm}, X_1, Y_1, Z_1, P_1)$$

while the unknown quantities (the components of the displacement vector, and also the temperature field variation function) can be sought in the form

$$Q(x, y, z, t) = \sqrt{\frac{2}{\pi}}\int_0^\infty [Q_1(x, y, z, \omega)\sin\omega t + Q_2(x, y, z, \omega)\cos\omega t]d\omega$$

These representations maintain in force all the formulae derived for the transforms of the required quantities. After solving the boundary-value problems it is necessary to revert to the originals of the required quantities.

Hence, the asymptotic method, used previously in Ref. 1–11 to solve non-classical boundary-value problems for anisotropic laminated plates and shells, can also be used to solve coupled dynamic problems of thermoelasticity, by first representing the unknown quantities by their Fourier transforms. The proposed asymptotic form and algorithm also enable one to solve coupled dynamic problems of thermoelasticity for anisotropic laminated non-uniform shells and plates of constant and variable thickness using the approach described earlier. Hence, the combination of the asymptotic method with the integral-transformation method extends the class of problems which can be effectively solved by the asymptotic method.

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